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REFUTING EHRENFEUCHT CONJECTURE ON RIGID MODELS†

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ABSTRACT
We prove that the class of cardinalities in which a first-order sentence has a rigid model can be very complicated, and we essentially characterize the possible classes.

Introduction

A rigid rigid model is a model with no automorphism except the identity.

Ehrenfeucht, in [2], builds some countable (first-order) theories T, to exemplify the possible $RS(T)=\{\lambda : T$ has a rigid model of cardinality $\lambda\}$. He conjectured that his examples exhaust, essentially, all posibilities; thus he conjectured that $RS(T)$ is always convex, its lower bound is $\aleph_0$ or $\aleph_1$, and its upper bound is ∞ or smaller than $\beth_{\omega_1}$. The positive solution of Łos conjecture by Morley [6] raised hopes, but no advance was made, except some works on some specific theories (order, Boolean algebras, see e.g. [5, 8, 11]). Ehrenfeucht conjecture is based on the thesis that "the only examples that exist are the ones that we can construct." Here we start from the dual thesis: "every class of cardinals can be represented as $RS(T)$, except when this is impossible on trivial grounds." More specifically, for every (first-order) sentence ψ, trivially $RS(\psi)$ is a Σ_2-class of pure second-order logic (i.e., there is a sentence

$$\chi = (\exists R_1, \cdots, R_n)(\forall S_1, \cdots, S_n)\,\theta,$$

θ first order, R_i, S_i variables over relations, and the class is $\{\lambda \mid \lambda \models \chi\}$). We prove an almost converse to this, i.e., restricting ourselves to $\lambda = \aleph_0 + \sum_{\aleph_0<\kappa<\lambda} \kappa^{\aleph_0}$. We can strengthen our result by weakening the condition on λ, but the effort seems too much for the gain.

An open problem is to remove the restriction on λ; but it is much more

† Dedicated to the memory of A. Robinson.

interesting to try to prove positive results for restricted classes, e.g. equational classes, or the class of $|T|^+$-saturated models. For sufficient conditions for the existence of rigid models, see Rubin and Shelah [7] and Shelah [10]. Ehrenfeucht conjecture appears in the problem lists of Chang and Keisler [1], and Friedman [3] and our solution was announced in [9].

NOTATIONS. λ, κ, μ denote infinite cardinals, $S_\omega(A)$ denotes the set of finite subsets of A. ${}^\omega a$, ${}^{\omega>}a$, ${}^{\omega\geq}a$ denote the set of functions from ω to a, $\{f \mid \mathrm{Dom}(f) \in \omega, \mathrm{Rng}(f) \subseteq a\}$ and ${}^\omega a \cup {}^{\omega>}a$, respectively; we shall use these notations also within models of a certain weak set theory.

If M is a model and R is a predicate in the language of M, then R^M denotes the interpretation of R in M; if R is unary, R^M also denotes the submodel of M whose universe is R^M. If $\varphi(x_1, \cdots, x_n)$ is a formula in the language of M then

$$\varphi^M = \{\langle a_1, \cdots, a_n \rangle \mid M \models \varphi[a_1, \cdots, a_n]\}.$$

If M is a model and $s_1, \cdots, s_l$ are relations or functions on $|M|$, then $(M, s_1, \cdots, s_l)$ denotes the model derived from M by adding names $s_1, \cdots, s_l$ to the language of M for the relations and functions $s_1, \cdots, s_l$. $M \overset{f}{\cong} N$ denotes that f is an isomorphism from M onto N. $\mathrm{Aut}(N)$ is the set of the automorphisms of N.

We shall use the conventions from set theory for denoting classes. Thus suppose $\varphi(x)$ is a formula with one free variable x. In the language, we denote $\{x \mid \varphi(x)\}$, say, by G; thus the formula $\varphi(x)$ is denoted by "$x \in G$", and in every model M, G^M denotes $\{a \mid M \models \varphi[a]\}$.

THEOREM 1. *For every Σ_2 sentence χ there exists a first order sentence ψ such that $RS(\psi) = \{\lambda \mid \lambda = \aleph_0$ or $\Sigma_{\kappa<\lambda}\kappa^{\aleph_0} = \lambda$; and $\lambda \models \chi\}$.*

Thus assuming G.C.H. for every Σ_2 sentence χ there exists a first-order sentence ψ such that $RS(\psi) = \{\lambda \mid \lambda \models \chi\}$.

Explanation of the proof

We try here to explain the proof. In the proof we define specific sentences and prove which are their models. Here we only describe the models. Note that in the proof, in each successive claim, we enlarge the model by adding new elements and relations. In order to simplify the explanation, we concentrate on the case $\mu = \mu^{\aleph_0}$, so we can get rid of the predicate R_1 and the parameter n, whose role is explained at the end.

The central part of the model N is P_0^N, which is a model of a small part of set theory. We want to assure that for rigid models, this part is:

i) with standard ω, hence standard finite sets,

ii) well-founded and even countably closed (i.e., all countable sets of elements of it are represented in it),

iii) the cardinality of N satisfies the Σ_2-sentence from Theorem 1.

We start with a model $M = M_1$ which will be P_0^N. In Claim 2 we enlarge the model so as to assure ω is standard. We look at ω^M with the successsor function, and add two copies of it, and the projection to the old ω^M. This adds the automorphism of interchanging the two copies. Moreover, ω^M and its copies are partitions naturally to components (two elements are in the same, if their "distance" is finite). Now for any set C of components, the function f which is the identity on M, and on the components not in C, and for each component in C, interchange its copies, is an automorphism of the enlarged model M_2. So it is enough to "kill" the automorphism interchanging the copies of the first component (and only them) as then, for rigid M, M_2 is rigid iff ω^M is standard. The "murder" is done by making the two copies of zero individual constants.

So from now on, in our set theory "finiteness" is standard.

In Claim 4 we eventually want to force our set theory to be countably closed. We first describe a similar construction assuring well-foundedness. This construction was used in a first version of the proof, but not here, and is already sufficient to disprove Ehrenfeucht conjecture on rigid models and a conjecture of Malitz in [4, p. 381] (the conjecture was that if a countable theory T has a rigid model of cardinality $\mu > \aleph_0$, then it has 2^μ non-isomorphic rigid models of cardinality μ).

Let G^k be the abelian group of order two generated freely by the decreasing sequences of ordinals (of M) of length k. More formally, the set of elements G^k is the family of finite sets consisting of decreasing sequences of ordinals of length k, and addition is interpreted as the symmetric difference. Notice that substruction for such groups is addition. We add to M_2 copies $G^{*k} = \{c_x : x \in G^k\}$ of G^k for each k, and almost give the natural isomorphism, so that the only information missing on G^{*k} is what is its zero. More exactly, instead of the natural isomorphism or addition, we add to M_2 the relation

$$Q = \{\langle c_x, c_y, z\rangle : \quad \text{for some } k \quad x, y, z \in G^k, x = y + z\}.$$

Every automorphism h of the new model over the old is determined by a sequence $\langle x(k) : k \in \omega^{M^1}\rangle$ where $x(k) \in G^{*k}$ and

$$h(a)=\begin{cases} a & a \in M_2 \\ c_{x+x(k)} & a = c_x, \qquad x \in G^k \end{cases}$$

Now we expand the model further by adding a function symbol for the projection Pr*, whose restriction to each G^{*k+1} is a homomorphism from G^{*k+1} into G^{*k} defined by $\mathrm{Pr}^*(\eta)=\eta \restriction k$, for η a decreasing sequence of ordinals of M of length $k+1$. Let M_3^* be the resulting model. It is easy to check that if $\langle x(k): k \in \omega^{M^1}\rangle$ determines an automorphism of M_3^* over M_2 then $\mathrm{Pr}^*(x_{k+1}) = x_k$. Let us show that when ω^M is standard, such a sequence of x_k's exists (except the sequence of zeros) iff there is an infinite decreasing sequence of ordinals of M.

If α_k is an infinite decreasing sequence of ordinals in M, $x_k = \langle \alpha_0, \cdots, \alpha_{k-1}\rangle$ will define a non-trivial automorphism of M. For the other direction remember x_k is a finite set of decreasing sequences of ordinals of M. If the automorphism is non-trivial for some k, $x_k \neq \emptyset$, and for each $\eta \in x_k$ there is $\eta_1 \in x_{k+1}$ such that $\mathrm{Pr}^*(\eta_1)=\eta$. So if $x_{k(0)} \neq \emptyset$ we can choose $\eta_{k(0)} \in x_{k(0)}$, and if $\eta_k \in x_k$ is defined, we can find $\eta_{k+1} \in x_{k+1}$, $\eta_{k+1} \restriction k = \eta_k$. So $\eta_{k(0)+n}(n)$ is an infinite decreasing sequence of ordinals in M. The construction we have described so far suffices to prove, e.g., that there is a sentence ψ such that ψ has a rigid model of cardinality μ iff μ is less than the first strongly inaccessible cardinal.

We now describe the construction that assures countable closedness. Let M_2 be the model which "took care" that ω will be standard. Let G^k be the abelian group of order 2 freely generated by sequences of length k from $P_0^{M_2}$. We first expand M_2 by adjoining to it for every $k \in \omega$ (in M_2) $G^{*k} \stackrel{\text{def}}{=} \{c_x \mid x \in G^k\}$ which is obviously a copy of G^k. Obviously we add the relation $\{\langle k, c_x\rangle \mid x \in G^k\}$ which is denoted by G^*. As in the previous case we add to M_2 the relation

$$Q = \{\langle c_x, c_y, z\rangle \mid \quad \text{for some} \quad k \quad x, y, z \in G^k \quad \text{and} \quad x = y + z\}$$

and the function Pr* which acts on each G^{*k+1} as the projection to G^{*k}. The resulting model, call it, say, M_2^1, has the following automorphisms: for every outer sequence η (that is a sequence that does not necessarily belong to $|M|$) of length ω^M there is a unique automorphism h_η of M_2^1 which is the identity on $P_0^{M_2^1}$ and such that $h_\eta(c_x) = c_{x+\eta\restriction k}$ for every $x \in G^k$. (M_2^1 has more automorphism, but it will be sufficient to "kill" those mentioned.) We shall expand M_2^1 to a new model M_3 so that h_η can be extended to an automorphism of M_3 iff η does not belong to $|M_2^1|$. For every $\eta \in |M_2^1|$ which is an ω-sequence in M_2^1 we adjoint to M_2^1 the set $\{c_{\eta,\sigma} \mid \sigma \in {}^M({}^{\omega>}\{0,1\})^M\}$ where ${}^{\omega>}\{0,1\}$ is the set of $\{0,1\}$-sequences of length ω which are eventually zero. Before defining the new relations let us first define $G_i^{k,\eta}$, $G_i^{*k,\eta}$ for $i \in \{0,1\}$. $G_0^{k,\eta}$ is the subgroup of G^k generated by all sequences of length k except $\eta \restriction k$; $G_1^{k,\eta} = \eta + G_0^k$ and $G_i^{*k,\eta} = \{c_x \mid x \in G_i^{k,\eta}\}$. Notice that $G_0^{k,\eta}$ is a subgroup of G^k of index 2 and $G_1^{k,\eta}$ is the other coset of $G_0^{k,\eta}$. We can now define the additional relation R_4:

$$R_4 = \{\langle \eta, c_{\eta,\sigma}, k, c_x\rangle \mid k \in \omega^M, \eta \in ({}^{\omega}P_0)^{M_2^1}, \sigma \in ({}^{\omega>}\{0,1\})^M \text{ and } x \in (G_{\eta(k)}^{k,\eta})^M\};$$

that is, $c_{\eta,\sigma}$ defines in each G^{*k} one of the cosets $G_0^{*k,\eta}$, namely $G_{\eta(k)}^{*k,\eta}$. If $\eta \in |M|$ then h_η cannot be extended to an automorphism of M_3, since h_η must take each $c_{\eta,\sigma}$ to c_{η,σ_1}, $\sigma_1(k) = 1 - \sigma(k)$, σ_1 is not eventually zero so there is no element c_{η,σ_1} in M_3. The reader can check that if h is a nontrivial automorphism of M_3 which is the identity on $|M_2|$ and ω^{M_3} is standard, then there is a sequence of elements of $P_0^{M_3}$ of length ω which is not in $|M_3|$ so M_3 is not countably closed.

In Claim 5 we try to assure that our model will satisfy the sentence from the theorem. We concentrate on a similar proof, from the first version, which does not appear here, and is suitable only for cardinals $\mu = \mu^{<\mu}$ (so assuming G.C.H., for regular cardinal $> \aleph_0$, we get the result). Under this assumption, similarly to Claim 4, we can assure M is isomorphic to the model $(H(\mu), \in)$ ($H(\mu)$—the family of sets of hereditary cardinality $< \mu$), so w.l.o.g. M is isomorphic to $(H(\mu), \in)$.

Let the sentence $(\exists \bar{R})(\forall \bar{S})\, \theta_1(\bar{R}, \bar{S})$ (θ_1—first order) and suppose μ satisfies it: then there is a sequence $\bar{R}$ of relations over μ such that $(\mu, \bar{R})$ fail to satisfy $(\exists \bar{S})\, \theta(\bar{R}, \bar{S})$ (θ—the negation of θ_1). In the model M we define a tree T of height μ: the elements of the α-th level will be elementary chains $\langle M_i^* : i < \alpha\rangle$, $M_i^* = (\beta_i, \bar{R} \restriction \beta_i, \bar{S}_i)$, M_i^* satisfies θ, $\beta_i \geqq i$. The order in the tree is continuation. It is clear that T has a branch of length μ iff $(\mu, \bar{R})$ satisfies $(\exists \bar{S})\, \theta(\bar{R}, \bar{S})$ (that is, $\bar{R}$ is not exemplifying the sentence).

We have to enlarge M_3 to M_4 such that M_4 has an automorphism over M_3 iff there is a branch of length μ in the tree T. This is not difficult: Let G_1^α be the free group of order two generated by the elements of T^α. We add to M_3 the sets of elements $G_1^{*\alpha} = \{c_x : x \in G^\alpha\}$, a relation similar to Q of Claim 4 and the partial three place function Pr_1^* such that for $\alpha < \beta < \mu$, $\mathrm{Pr}_1^*(\alpha, \beta, x)$ is the natural projection from $G_1^{*\beta}$ into $G_1^{*\alpha}$. Each automorphism h of M_4 over M_3 corresponds to a sequence $\langle x_\alpha : \alpha < \mu\rangle\, x_\alpha \in G_1^\alpha$, $\mathrm{Pr}_1(\alpha, \beta, x_\beta) = x_\alpha$ ($\mathrm{Pr}_1(\alpha, \beta, x)$ is the natural projection from G_1^β into G_1^α).

Note that if $\{t_\alpha : \alpha < \mu\}, (t_\alpha \in T_\alpha)$ is a branch, taking t_α for x_α gives us a non-trivial automorphism. On the other hand, suppose h, x_α are given, h

non-trivial. Note x_α is a finite subset of T_α, and by the condition above $|x_\alpha| \leqq |x_\beta|$ for $\alpha < \beta$. Hence for some $n_0 < \omega$, $\alpha_0 < \mu$, for every $\beta \geqq \alpha_0$ $|x_\beta| = n_0$. By the non-triviality of h, $n_0 \neq 0$. It follows that when $\alpha_0 < \alpha < \beta$, $\mathrm{Pr}_1(\alpha, \beta, x)$ induces a one-to-one function from x_β onto x_α. Remembering that for $\alpha < \beta < \gamma$, $x \in G_1^\gamma$, $\mathrm{Pr}_1(\alpha, \gamma, x) = \mathrm{Pr}_1(\alpha, \beta, \mathrm{Pr}_1(\beta, \gamma, x))$, we can find $t_\alpha \in x_\alpha$ for $\alpha \geqq \alpha_0$ such that $\mathrm{Pr}_1(\alpha, \beta, t_\beta) = t_\alpha$ for $\alpha_0 < \alpha < \beta < \mu$. Clearly $\{t$: for some $\alpha > \alpha_0, t < t_\alpha\}$ is a branch. This completes the proof of the revised Claim 5.

We still have to explain the proof of the actual Claim 5. Notice that the tree T was a set of approximations $\bar{S}$ such that $(\mu, \bar{R}, \bar{S})$ satisfies θ. In the proof of Claim 5 itself we shall use only countable approximations (which by Claim 4 are standard). For natural reasons we do not elaborate.

We still have a small debt — the cases $\mu < \mu^{\aleph_0}$. We ignore the case $\lambda = \aleph_0$, so $\mu = \Sigma_{n<\omega}\mu_n$, $\mu_n < \mu$, $\mu_n^{\aleph_0} = \mu_n$. In this case it is impossible to demand M is countably closed. Hence we represent μ as an increasing union of length ω of transitive submodels, which are countably closed. $R_1(n, x)$ will say that x is in the n-th model. This will complicate somewhat all the proof.

Remark. For brevity we deal only with $\lambda > \aleph_0$, however, the case $\lambda = \aleph_0$ is easier.

Proof. The proof is divided into subclaims.

Let $L_1 = \{\in, =, R_1\}$, where $\in, R_1$ are 2-place predicates.

Let ψ_1 be the sentence which says:

1) $\in$ is extensional.

2) $\in$ is "well founded", that is

$$\forall x(\exists y(y \in x) \rightarrow \exists y(y \in x \wedge \forall z) \neg(z \in y \wedge z \in x)).$$

3) $\in$ satisfies the pair axiom, that is, for every x and y there exists $\{x, y\}$.

4) $\in$ satisfies the union set axiom, and the (set theoretic) difference between any two sets exists.

5) $\in$ satisfies the intersection set axiom, that is for every a there exists $\cap a$.

6) For every a, b $a \times b$ exists.

7) If f is a function and a is a set then $\{f(b): b \in a\}$ exists.

8) If f is a function and a is a set then $f \restriction a$ exists. a is an ordinal if $\in$ linearly orders a, and a is transitive.

9) There exists a first limit ordinal; we denote it by ω.

10) $\forall x \forall y(R_1(x, y) \rightarrow x \in \omega)$. Let $R_1^x = \{y \mid R_1(x, y)\}$.

11) $\forall x \forall y (x \in y \in \omega \rightarrow R_1^x \subseteq R_1^y)$.

12) $\forall y \exists x R_1(x, y)$, and for every x R_1^x is a transitive set.

13) $\omega \subseteq R_1^0$ where $0 = \emptyset$ is the first element in ω.

Let $L_2 = L_1 \cup \{P_0, c^0, c^1, R_2, R_3\}$ where P_0, R_2, R_3 are 1-place, 2-place, 2-place relations, respectively, and c^0 and c^1 are individual constants.

For every $M \models \psi_1$ we define a model $N = N_2(M)$ in the language $L_2 : P_0^N = M$, $|N| - |M| = \{c_x^i \mid i = 0,1 \text{ and } x \in \omega^M\}$, $(c^i)^N = c_0^i$, $i = 0, 1$, $R_3^N = \{\langle c_x^i, c_{x+1}^i \rangle \mid i = 0,1 \text{ and } x \in \omega^M\}$ and $R_2^N = \{\langle x, c_x^i \rangle \mid x \in \omega^M \text{ and } i = 0, 1\}$.

CLAIM 2. There exists a sentence ψ_2 in the language L_2 such that:

a) For every $M \models \psi_1$, $N_2(M) \models \psi_2$.

b) If $M \models \psi_2$ then $P_0^N \models \psi_1$ and there is an f such that $N \overset{f}{\cong} N_2(P_0^N)$ and $f \restriction P_0^N = \mathrm{Id}$.

c) If $M \models \psi_1$ then ω^M is not standard iff there is an $f \in \mathrm{Aut}(N_2(M))$ such that $f \neq \mathrm{Id}$ and $f \restriction |M| = \mathrm{Id}$.

PROOF. It is easy to find a sentence ψ_2 which satisfies (a) and (b). We prove (c). Let $N \models \psi_2$. By (b) we can assume that $N = N_2(M)$ for some M such that $M \models \psi_1$. Suppose ω^M is not standard, then: $\langle \omega^M, \in \rangle \cong \omega + Z\sigma$. Let $f : |N| \rightarrow |N|$ be defined as follows: $f \restriction |M| = \mathrm{Id}$, if $a \in |N| - |M|$ and a is the n-th successor of $(c^i)^N$, $i = 0, 1$ (that is, there are $(c^i)^N = a_1, \cdots, a_n = a$ such that $\langle a_i, a_{i+1} \rangle \in R_3^N$) then $f(a) = a$; otherwise there exist b, c such that $b \neq a$ and $\langle c, a \rangle, \langle c, b \rangle \in R_2^N$, we define $f(a) = b$. It is easy to check that f is a non-trivial automorphism of N and $f \restriction |M| = \mathrm{Id}$.

DEFINITION 3. Let $M \models \psi_1$. We say that M is countably closed if for every $x \in \omega^M$ and for every $a \subseteq (R_1^x)^M$ such that $|a| = \aleph_0$, there exists $a' \in M$ such that for every x, $x \in^M a'$ iff $x \in a$.

In the sequel we shall define a language $L_3 \supseteq L_2 \cup \{P_1\}$, a sentence ψ_3 in L_3 and a function $M \mapsto N_3(M)$ such that for every $M \models \psi_1$, $N_3(M)$ is a model in L_3 and the following claim is satisfied:

CLAIM 4. $N_3(M)$, ψ_3, L_3 satisfy the following:

a) If $M \models \psi_1$ then $N_3(M) \models \psi_3$ and if $N \models \psi_3$ then $P_0^N \models \psi_1$.

b) If $M \models \psi_1$ then $P_0^{N_3(M)} \restriction L_1 = M$; if $N \models \psi_3$ then there exists an f such that $N_2(P_0^N) \overset{f}{\cong} P_1^N \restriction L_2$ and $f \restriction P_0^N = \mathrm{Id}$.

c) Let $N \models \psi_3$ and $M = P_0^N$ then there exists an $\mathrm{Id} \neq f \in \mathrm{Aut}(N)$ such that $f \restriction (|N| - (P_1^N - P_0^N)) = \mathrm{Id}$ iff ω^M is not standard.

d) Let $N \models \psi_3$ then:

i) If P_0^N is not countably closed then there is an $\mathrm{Id} \neq f \in \mathrm{Aut}(N)$ such that $f \restriction P_0^N = \mathrm{Id}$.

ii) If P_0^N is countably closed (and therefore well-founded) then N is rigid.

e) If $N \models \psi_3$ then $|P_0^N| = \|N\|$.

PROOF. We first define the following abbreviations for every $n, k \in \omega$

$$\langle G^{n,k}, + \rangle = \langle S_\omega({}^k(R_1^n)), \text{ symmetric difference} \rangle,$$

that is, $x \in G^{n,k}$ is the abbreviation of the formula of L_2 which says that $n, k \in \omega$, x is a finite set of functions from k to R_1^n and $x + y = z$ means that $x, y, z \in G^{n,k}$ and $(x - y) \cup (y - x) = z$.

For notational simplicity let distinct groups $G^{n,k}$ have distinct zeros. Note that in those groups every element has order two, hence addition and subtraction are the same. For every $n, k \in \omega$ there is a unique homomorphism $\mathrm{Pr}\colon G^{n,k+1} \to G^{n,k}$ such that for every $x \in {}^{k+1}(R_1^n)$ $\mathrm{Pr}(\{x\}) = \{x \restriction k\}$. $\mathrm{Pr}(x) = y$ can be defined in L_2.

Let $L_3 = L_2 \cup \{P_1, P_2, G^*, Q, \mathrm{Pr}^*, P_3, R_4\}$ where $P_1, P_2, G^*, Q, P_3, R_4$ are 1-place, 2-place, 3-place, 3-place, 3-place and 5-place predicates, respectively, and Pr^* is a unary function symbol. We shall use the following notations: $P_2^n = \{x \mid P_2(n, x)\}$, $P_3^{n,\eta} = \{x \mid P_3(n, \eta, x)\}$, $G^{*n,k} = \{x \mid G^*(n, k, x)\}$, $R_4^{n,\eta,x,k} = \{y \mid R_4(n, \eta, x, k, y)\}$. Let us define $N_3(M)$ for M's which satisfy ψ_1; we denote $N = N_3(M)$. Let $P_1^N = N_2(M)$, for every $n, k \in \omega^M$ $\{c_{x,n} \mid x \in (G^{n,k})^M\} = (G^{*n,k})^N$, that is, the elements of $(G^{*n,k})^N$ are copies of the elements of $(G^{n,k})^M$ and for $\langle n, k \rangle \neq \langle n_1, k_1 \rangle$ $(G^{*n,k})^N \cap (G^{*n_1,k_1})^N = \emptyset$; if $\langle x, y \rangle \notin (\omega^M)^2$ then $(G^{*x,y})^N = \emptyset$. For every $n \in \omega^M$, $(P_2^n)^N = \bigcup_{k \in \omega^M} (G^{*n,k})^N$. For every $n \in \omega^M$ and $\eta \in ({}^\omega(R_1^n))^M$

$$(P_3^{n,\eta})^N = \{c_{n,\eta,\sigma} \mid \sigma \in ({}^{\omega>}\{0,1\})^M\}$$

$(({}^{\omega>}\{0,1\})^M)$ is the set of those elements of M which are sequences 0, 1 of length ω which are eventually zero). Let $|N| = P_1^N \cup \bigcup_{n \in \omega^M} (P_2^n)^N \cup \bigcup \{(P_3^{n,\eta})^N \mid n, \eta \in |M|\}$;

$$Q^N = \left\{ \langle c_{x_1,n}, c_{x_2,n}, y \rangle \;\middle|\; \begin{array}{l} n \in \omega^M \text{ and there is } k \text{ such that} \\ x_1, x_2 \in (G^{n,k})^M \text{ and } x_1 + x_2 = y \end{array} \right\}$$

$\mathrm{Pr}^{*N}(c_{x,n}) = c_{\mathrm{Pr}(x),n}$ and if y is not of the form $c_{x,n}$ then $\mathrm{Pr}^{*N}(y) = \emptyset^M$. Before defining R_4 let us define the formulas $x \in G_0^{n,k,\eta}$, $x \in G_1^{n,k,\eta}$ in the language of M. If $\eta \in {}^{\omega\geq}(R_1^n)$ and $k \leq l(\eta)$ (the length of η), $G_0^{n,k,\eta}$ is the subgroup of $G^{n,k}$ generated by $\{\{t\} \mid t \in {}^k(R_1^n)\} - \{\{\eta \restriction k\}\}$ and $G_1^{n,k,\eta} = G_0^{n,k,\eta} + \{\eta \restriction k\}$.

$$R_4^N = \{\langle n, \eta, c_{n,\eta,\sigma}, k, c_{x,n}\rangle \mid n, k \in \omega^M, \eta \in ({}^{\omega}(R_1^n))^M, \sigma \in ({}^{\omega>}\{0,1\})^M \quad \text{and} \quad x \in (G_{\sigma(k)}^{n,k,\eta})^M\}.$$

So if $(R_4^{n,\eta,x,k})^N \neq \emptyset$ then it is either the copy of $G_0^{n,k,\eta}$ or the copy of $G_1^{n,k,\eta}$.

Let ψ_3 be the sentence in L_3 which says:

1) ψ_3', where ψ_3' is the sentence which tells all the information on the domains of all the predicates in L_3; for instance ψ_3' says:

$\forall y \forall x (P_2(y,x) \rightarrow y \in \omega \wedge (\neg P_1(x)),$

$$P_2^n \models \bigcup_{k\in\omega} G^{*n,k}, \forall x \forall y (x \in y \rightarrow P_0(x) \wedge P_0(y)),$$

$$\forall n_1, n_2, k_1, k_2 (\langle n_1, k_1\rangle \neq \langle n_2, k_2\rangle \rightarrow G^{*n_1,k_1} \cap G^{*n_2,k_2} = \emptyset)$$

$\forall x, y, z (Q(x,y,z) \rightarrow$ there are $n, k \in \omega$ such that $z \in G^{n,k}$ and $x, y \in G^{*n,k})$, etc.

2) $\psi_2^{P_1}$

Intuitively $Q(x,y,z)$ will say that x, y are copies of x', y' such that $x', y', z \in G^{n,k}$ and $x' + y' = z$.

3) (i) For every $x, y \in G^{*n,k}$ there is a unique z such that $Q(x,y,z)$;

(ii) For every $y \in G^{*n,k}$ and $z \in G^{n,k}$ there is a unique x such that $Q(x,y,z)$; for a fixed y (or x) we get a one-to-one correspondence between $G^{*n,k}$ and $G^{n,k}$;

(iii) $\forall x, y, z (Q(x,y,z) \rightarrow x = y \equiv z = 0)$;

(iv) $\forall x, y, z (Q(x,y,z) \rightarrow Q(\mathrm{Pr}^*(x), \mathrm{Pr}^*(y), \mathrm{Pr}(z))$;

(v) $\forall x_1, y_1, x_2, y_2, z (Q(x_1,y_1,z) \wedge Q(x_2,y_2,z) \rightarrow \exists u (Q(x_1,x_2,u) \wedge Q(y_1,y_2,u))$ (that is, $x_1 + y_1 = x_2 + y_2 \rightarrow x_1 + x_2 = y_1 + y_2$);

(vi) $\forall x, y, z, u, v((Q(x,y,u) \wedge Q(y,z,v)) \rightarrow Q(x,z,u+v))$ (that is, $(x+y)+(y+z) = x+z$).

Let $xE_\eta^{n,k}y$ be the formula which says that $n, k \in \omega$, $x, y \in G^{*n,k}$, $\eta \in {}^{\omega\geq}(R_1^n)$, $l(\eta) \geq k$ and if $Q(x,y,z)$ then $z \in G_0^{n,k,\eta}$. From 3) (vi) it follows that $E_\eta^{n,k}$ is an equivalence relation on $G^{*n,k}$, with exactly two equivalence classes.

4) (i) $\forall n \in \omega)(\forall \eta \in {}^{\omega}(R_1^n))(\forall x \in P_3^{n,\eta})(\forall k \in \omega)(R_4^{n,\eta,x,k}$ is an equivalence class of $E_\eta^{n,k})$.

(ii) $(\forall n \in \omega)(\forall \eta \in {}^{\omega}(R_1^n))(\forall x, y \in P_3^{n,\eta})[(\forall k \in \omega) R_4^{n,\eta,x,k} = R_4^{n,\eta,y,k} \rightarrow x = y)]$.

Let

$$\varphi(n,\eta,x,y,\sigma) \equiv (n \in \omega) \wedge (\eta \in {}^{\omega}(R_1^n)) \wedge (x, y \in P_3^{n,\eta}) \wedge \sigma \in {}^{\omega}\{0,1\} \wedge (\forall k \in \omega)$$
$$(R_4^{n,\eta,x,k} = R_4^{n,\eta,y,k} \equiv \sigma(k) = 0).$$

(iii) $(\forall n \in \omega)(\forall \eta \in {}^{\omega}(R_1^n))(\forall x, y \in P_3^{n,\eta}) \exists \sigma \varphi(n,\eta,x,y,\sigma)$.

(iv) $(\forall n \in \omega)(\forall \eta \in {}^{\omega}(R_1^n)(\forall x \in P_3^{n,\eta})(\forall \sigma \in {}^{\omega>}\{0,1\}) \exists y \varphi(n,\eta,x,y,\sigma)$.

It is easy to see that ψ_3 and $N_3(M)$ satisfy (a), (b) and (e) of Claim 4.

c) Let $N \models \psi_3, M_1 = P_1^N, M_0 = P_0^N$. By (2) $M_1 \models \psi_2$, by (a) and (1) M_1 is isomorphic over M_0 to $N_2(M_0)$. So if ω^{M_0} is standard then by Claim 2 there is no non-trivial automorphism of M_1, which is the identity on M_0; since $|M_1|$ is definable in N, there is no non-trivial automorphism of N such that $f \restriction (|N| - (|M_1| - |M_0|)) = \mathrm{Id}$.

If ω^{M_0} is not standard then there is an $f \in \mathrm{Aut}(M_1)$ such that $f \restriction M_0 = \mathrm{Id}$; by the axioms in (1) and since $f \restriction M_0 = \mathrm{Id}$, $f \cup \mathrm{Id} \restriction (|N| - |M_1|)$ is an automorphism of N.

d) Let $N \models \psi_3$, M_0, M_1, M_2 are the submodels of N whose universes are P_0^N, P_1^N, $P_1^N \cup \bigcup \{(P_2^n)^N \mid n \in \omega^N\}$ respectively. We first show that if $f\colon |M_2| \cong |M_2|$ satisfies $f \restriction |M_1| = \mathrm{Id}$ then $f \in \mathrm{Aut}(M_2)$ iff for every $n \in \omega^N$: for every $k \in \omega^N$ there is an $a_k \in G^{n,k,N}$ such that for every $b \in G^{*n,k,N}$ $\langle f(b), b, a_k \rangle \in Q^N$, and $\mathrm{Pr}^N(a_{k+1}) = a_k$.

Let $b, c \in (G^{*,n,k})^N$, then $\langle b, c, d \rangle \in Q^N \Rightarrow \langle f(b), f(c), d \rangle \in Q^N \Rightarrow$ by (3) (v) there is an a such that $\langle f(b), b, a \rangle, \langle f(c), c, a \rangle \in Q^N$; by (3) (i) $a = a_{n,k}$ does not depend on b or c. As $\langle f(b), b, a_{n,k} \rangle \in Q^N$, $\langle \mathrm{Pr}^{*N}(f(b)), \mathrm{Pr}^{*N}(b), \mathrm{Pr}^N(a_{n,k}) \rangle \in Q^N$ and since $f \in \mathrm{Aut}(M_2)$ $\mathrm{Pr}^{*N}(f(b)) = f(\mathrm{Pr}^{*N}(b))$ so $c = \mathrm{Pr}^{*N}(f(b)) \in (G^{*n,k-1})^N$, and $\langle f(c), c, \mathrm{Pr}^N(a_{n,k}) \rangle \in Q^N$ so $\mathrm{Pr}^N(a_{n,k}) = a_{n,k-1}$.

d) (i) Suppose $M_0 = P_0^N$ is not countably closed; if also ω^N is not standard then by (c) there is $\mathrm{Id} \neq f \in \mathrm{Aut}(N)$ such that $f \restriction |M_0| = \mathrm{Id}$. Suppose ω^N is standard and let $a \subseteq (R_1^n)^N$ $|a| = \aleph_0$ and $a \not\in N$ (that is, for no $a' \in |N|$ $\forall x (x \in^N a' \equiv x \in a)$). Let η_0 be a function from ω onto a, then by axiom 7 of ψ_1 $\eta_0 \not\in N$. Let f be defined as follows: $f \restriction (|N| - (P_2^n)^N - \{P_3^{n,\eta,N} \mid \eta \in |N|\}) = \mathrm{Id}$, if $x \in G^{*n,k}$ then $f(x)$ is the unique y which satisfies $\langle y, x, \{\eta_0 \restriction k\} \rangle \in Q^N$ (notice that ψ_1 assures that N is closed under finite sequences); let $\eta \in ({}^{\omega}(R_1^n))^N$, $k \in \omega^N$, $\eta \restriction k = \eta_0 \restriction k$, and $\eta \restriction (k+1) \neq \eta_0 \restriction (k+1)$. Let $l(\sigma) = k$, $\sigma = \langle 1, \cdots, 1 \rangle$ for every $x \in (P_3^{n,\eta})^N$ we define $f(x)$ to be the unique y which satisfies $N \models \varphi(n, \eta, x, y, \sigma)$ (see axiom (4) (iii)). We identify σ with $\sigma^\frown\langle 0, 0, \cdots \rangle$.

By the first claim in the proof of (d) $f \restriction |M_2| \in \mathrm{Aut}(M_2)$. By the definition of f $\langle n, \eta, x \rangle \in P_3^N$ iff $\langle f(n), f(\eta), f(x) \rangle = \langle n, \eta, f(x) \rangle \in P_3^N$. Suppose $\langle n, \eta, x, k, y \rangle \in R_4^N$, let $k_0 \in \omega^N$ such that $\eta \restriction k_0 = \eta_0 \restriction k_0$ and $\eta \restriction k_0 + 1 \neq \eta_0 \restriction k_0 + 1$. If $k > k_0$ then by the definition of f $(R_4^{n,\eta,x,k})^N = (R_4^{n,\eta,f(x),k})^N$, since $\langle f(y), y, \{\eta_0 \restriction k\} \rangle \in Q^N$ and $\{\eta_0 \restriction k\} \in (G_0^{n,k,\eta})^N$, $f(y) E_\eta^{n,k} y$; thus by 4 (i) $f(y) \in (R_4^{n,\eta,x,k})^N = R_4$ so $\langle n, \eta, f(x), k, f(y) \rangle \in R_4^N$. If $k \leqq k_0$ then $(R_4^{n,\eta,x,k})^N \neq (R_4^{n,\eta,f(x),k})^N$ and $\neg f(y) E_\eta^{n,k} y$ so again $\langle n, \eta, f(x), k, f(y) \rangle \in R_4^N$, since by axioms (3) (iii) and (3) (vi) $f^2 = \mathrm{Id}$ it follows that $\langle f(n), f(\eta), f(x), f(k), f(y) \rangle \in R_4^N \Leftrightarrow \langle n, \eta, x, k, y \rangle \in R_4^N$.

d) (ii) Let $N \models \psi_3$ and $\mathrm{Aut}(N) \neq \mathrm{Id}$. Suppose M_0 is well founded; we show that M_0 is not countably closed. let $\mathrm{Id} \neq f \in \mathrm{Aut}(N)$, since M_0 is well founded $f \upharpoonright M_1 = \mathrm{Id}$. It is clear that for some $n \in \omega^N f \upharpoonright (P_2^n)^N \neq \mathrm{Id}$. Let $\{a_k \mid k \in \omega^N\}$ be as in the first claim in the proof of (d). Since $f \neq \mathrm{Id}$ for some $k_0 \in \omega^N$, for every $k \geqq k_0$ $a_k \neq \emptyset$. From the definition of Pr there is a sequence $\{z_k \mid k_0 \leqq k \in \omega^N\}$ such that $z_k \in^N a_k$ and for every $k \in \omega^N$ $z_k = z_{k+1} \upharpoonright k$. Let $\eta = \{z_k \mid k \in \omega^N\}$. Suppose by way of contradiction $\eta \in |N|$. Then for every $k_0 \leqq k \in \omega^N$ and $x \in (G^{*n,k})^N$ not $x E_\eta^{n,k} f(x)$. Let $y \in P_3^{n,\eta}$ then for some $k_0 < k$ $(R_4^{n,\eta,y,k})^N = (R^{n,\eta,f(y),k})^N$ so $\langle n, \eta, y, k, x\rangle \in R_4^N \Leftrightarrow \langle n, \eta, f(y), k, f(x)\rangle \notin R_4^N$, a contradiction, so $\eta \notin |N|$ and thus N is not countably closed. So Claim 4 is proven.

* * *

Let $\chi_2 = \exists R_1^1 \cdots \exists R_{n_1}^1 \forall S_1 \cdots \forall S_k \chi_3$ be a second-order sentence and χ_3 a first-order sentence, then χ_2 is logically equivalent to a sentence

$$\chi \equiv \exists R_1^1 \cdots \exists R_{n_1}^1 \neg \exists S_1^1 \cdots \exists S_l^1 \forall x_1 \cdots \forall x_{m_1} \chi_1(R_1^1 \cdots R_{n_1}^1, S_1^1 \cdots S_l^1, x_1, \cdots, x_{m_1})$$

where $S_1^1 \cdots S_l^1$ are relations or function symbols and χ_1 is quantifier free. Let $\chi_4 \equiv \neg \exists S_1^1 \cdots S_l^1 \forall x_1 \cdots \forall x_m \chi_1$; we denote $\bar{R} = \langle R_1^1 \cdots R_{n_1}^1\rangle$ and $\bar{S} = \langle S_1^1 \cdots S_l^1\rangle$. W.l.o.g. for every j S_j^1 is a k_j-place function symbol.

We shall define $L_4 \supseteq L_3 \cup \{R_1^1, \cdots, R_{n_1}^1\}$, a sentence ψ_4 in L_4 and for every model M in the language $L_1^* = L_1 \cup \{R_1^1, \cdots, R_{n_1}^1\}$ which satisfies ψ_1, a model $N_4(M)$ such that the following claim holds:

Claim 5. i) If $M \models \psi_1$ and $L(M) = L_1^*$ then $N_4(M) \models \psi_4$ and if $N \models \psi_4$ then $P_0^N \models \psi_1$.

ii) If $N \models \psi_4$ then $\|N\| = |P_0^N|$.

iii) If $N \models \psi_4$, $M = P_0^N$, M is countably closed and $M \models \psi_1 \wedge \chi_4$ then N is rigid.

iv) If $N \models \psi_4$ and N is rigid then $P_0^N \models \psi_1 \wedge \chi_4$ and P_0^N is countably closed.

Proof. Certain details in the proof of Claim 5 are similar to what was done in Claim 4; we omit these details.

We first define certain formulas in L_1^*. There is a formula $\varphi(n, a, \bar{s})$ in L_1^* such that for every model M of ψ_1 in the language L_1^*, for every $n, a \in |M|$ and $\bar{s} = \langle s_1, \cdots, s_l\rangle \in |M|$, $M \models \varphi[n, a, \bar{s}]$ iff the following holds in $M: n \in \omega^M$, $a \subseteq R_1^n$, a is countable (in M), s_i is a function, $\mathrm{Dom}(s_i) \subseteq a^{k_i}$, $\mathrm{Rng}(s_i) \subseteq R_1^n$, $i = 1, \cdots, l$ and for every $x_1, \cdots, x_{m_1} \in a$ if the truth value of $\chi_1(R_1^1, \cdots, R_{n_1}^1, s_1, \cdots, s_l, x_1, \cdots, x_{m_1})$ is defined then it is "truth".

Let k be a (true) natural number. We say that $\langle s, s'\rangle$ is an n-approximation on a^k ($a^k = a \times a \times \cdots \times a$ of length k) if $a \subseteq R_1^n$, s and s' are functions, $\mathrm{Dom}(s) \cap \mathrm{Dom}(s') = \emptyset$, $\mathrm{Dom}(s) \cup \mathrm{Dom}(s') = a^k$, $\mathrm{Rng}(s) \subseteq R_1^n$, $\mathrm{Rng}(s') \subseteq$

$\{m : n < m \subseteq \omega\}$; we say that $\langle s, s' \rangle$ is the n-approximation induced on a^k by f if $\langle s, s' \rangle$ is an n-approximation on a^k, $s = f \restriction (a^k \cap f^{-1}(R_1^n))$ and $s'(\bar{x}) = m$ iff $f(\bar{x}) \in R_1^m - R_1^{m-1}$.

We now define $G_1^{a,n}$: $G_1^{a,n} = S_\omega(\{\langle\langle s_1, s_1'\rangle, \cdots, \langle s_l, s_l'\rangle\rangle \mid \varphi(n, a, \langle s_1, \cdots, s_l\rangle)$ and $\langle s_i, s_i'\rangle$ is an n-approximation on a^{k_i}, $i = 1, \cdots, l\})$; $+$ denotes the symmetric difference in $G_1^{a,n}$. We define now the function Pr_1. Let $b \subseteq a$, $m \leqq n$, $\langle s_1, s_1'\rangle$ is an n-approximation on a^k. Let $s_1^* = s_1 \restriction (b^k \cap s_1^{-1}((R_1^m)^k)$ and $s_1'^* = s_1' \restriction b^k \cup s''$, where $s''(\bar{x}) = r$ iff $\bar{x} \in b^k \cap \mathrm{Dom}(s_1)$ and $s_1(\bar{x}) \in R_1^r - R_1^{r-1}$ and $r > m$ and $\langle s_1, s_1'\rangle^* = \langle s_1^*, s_1'^*\rangle$.

$$\mathrm{Pr}_1(a, n, b, m, x) = \begin{cases} \{\langle \bar{s}^{1*}, \cdots, \bar{s}_l^{1*}\rangle, \cdots, \langle \bar{s}_1^{r*}, \cdots, \bar{s}_l^{r*}\rangle\} & \text{if: } x \in G^{a,n}\, m \leqq n,\ b \subseteq a, \text{ and let } x = \{\langle \bar{s}_1^1, \cdots, \bar{s}_l^1\rangle, \cdots, \langle \bar{s}_1^r, \cdots, \bar{s}_l^r\rangle\} \\ \emptyset & \text{otherwise.} \end{cases}$$

Certainly there are formulas in L_1^* which define in every model of ψ_1 in L_1^* the relations $x \in G_1^{a,n}$ and $y = \mathrm{Pr}_1(a, n, b, m, x)$.

Let $L_4 = L_3 \cup \{R_1^1, \cdots, R_{n_1}^1\} \cup \{P_4, G_1^*, \mathrm{Pr}_1^*, Q_1\}$ where P_4, G_1^*, Q_1 are 1-place, 3-place, and 3-place predicates, respectively, and Pr_1^* is a 5-place function symbol.

Let M be a model of ψ_1 in the language L_1^*. We define $N = N_4(M) : |N| = |N_3(M)| \cup \{c_{a,n,t} \mid t \in (G_1^{a,n})^M\}$, $P_4^N = N_3(M)$, $(G_1^*)^N = \{\langle a, n, c_{a,n,t}\rangle \mid t \in (G^{a,n})^M\}$, $Q_1^N = \{\langle c_{a,n,t_1}, c_{a,n,t_2}, t_1 - t_2\rangle \mid t_1, t_2 \in (G_1^{a,n})^M\}$, the relations and functions of $N_3(M)$, and

$$\mathrm{Pr}^{*N}_1(a, n, b, m, x) = \begin{cases} c_{b,m,d} & \text{if } x = c_{a,n,t} \text{ and in } M \quad m \leqq n, \quad b \subseteq a, \quad t \in G_1^{a,n}, \quad G_1^{b,m} \neq \emptyset, \\ & \qquad d = \mathrm{Pr}_1^M(a, n, b, m, t) \\ \emptyset & \text{otherwise.} \end{cases}$$

ψ_4 is defined in a similar way to the definition of ψ_3.

The proof of (i), (ii) in Claim 5 is trivial. To prove (iv) suppose $N \vDash \psi_4$ is rigid and denote $M = P_0^N$. Certainly M is countably closed. Suppose by way of contradiction $M \nvDash \chi_4$; then there are functions $s_1, \cdots, s_l$ on $|M|$ such that $(M, s_1, \cdots, s_l) \vDash \forall x_1 \cdots \forall x_{m_1} \chi_1(\bar{R}, \bar{s}, x_1, \cdots, x_{m_1})$. For every $a, n \in |M|$ such that $(G_1^{a,n})^M \neq \emptyset$, let $t_{a,n} = \{\langle \bar{s}_1^*, \cdots, \bar{s}_l^*\rangle\}$ where $\bar{s}_i^*$ is the n-approximation induced on a by s_i. We define $f : |N| \to |N|$:

$$f(x)=\begin{cases} x & \text{if } x\in |N_3(M)| \\ y & \begin{cases}\text{if } x\notin M \text{ and } y \text{ is the unique element such that for some} \\ a,n\in|M|, \langle y,x,t_{a,n}\rangle\in Q_1^N\end{cases}\end{cases}$$

as in Claim 4 it is easy to see that f is a nontrivial automorphism of N, a contradiction.

iii) Suppose by way of contradiction N is not rigid, let $M=P_0^N$ and $\mathrm{Id}\neq f\in \mathrm{Aut}(N)$: Since M is well founded $f\restriction|M|=\mathrm{Id}$; since M is countably closed and $P_4^N\models\psi_3$ (this can be assured by ψ_4) by (a) (ii) of Claim 4 $f\restriction P_4^N=\mathrm{Id}$. As in the analogous part in the proof of Claim 4 for every $a,n\in|M|$ such that $(G_1^{a,n})^M\neq\emptyset$ there is a unique $t_{a,n}\in(G_1^{a,n})^N$ such that for every $x\in(G_1^{*a,n})^N$ $\langle f(x),x,t_{a,n}\rangle\in Q_1^N$, $\mathrm{Pr}_1^N(a,n,b,m,t_{a,n})=t_{b,m}$ and for some a',n' $t_{a',n'}\neq\emptyset$. So it is easy to define for every $n'\leqq n\in\omega^M$ a_n such that

1) $(G^{a_n,n})^M\neq\emptyset$.

2) If $m>n$, $a'\subseteq a_n\subseteq a_m$.

3) If $a\subseteq b$ and $(G^{b,n})^M\neq\emptyset$ then $|t_{b,n}|=|t_{a_n,n}|$. Note that if $\mathrm{Pr}_1^N(a,n,b,n,t_1)=t_2$ then $|t_2|\leqq|t_1|$. It is possible to choose a sequence $\{\overline{s^n}\mid n'\leqq n\in\omega\}$ such that $\overline{s^n}\in t_{a_n,n}$ and $m<n\Rightarrow \mathrm{Pr}_1^M(a_n,n,a_m,m,\{\overline{s^n}\})=\overline{s^m}$. For every b and $n\geqq n'$ such that $(G^{b,n})^M\neq\emptyset$ and $b\supseteq a_n$ there is a unique $\overline{s^{b,n}}\in t_{b,n}$ such that $\mathrm{Pr}_1^M(b,n,a_n,n,\overline{s^{b,n}})=\overline{s^n}$. We show that if $b\subseteq c$, $m\leqq k$ $(G_1^{c,k})^M\neq\emptyset\neq(G_1^{b,m})^M$ then $\mathrm{Pr}_1^M(c,k,b,m,\overline{s^{c,k}})=\overline{s^{b,m}}$. Let $\bar{s}=\mathrm{Pr}_1^M(c,k,b,m,\overline{s^{c,k}})$ then

$$\begin{aligned}\mathrm{Pr}_1^M(b,m,a_m,m,\bar{s})&=\mathrm{Pr}_1^M(b,m,a_m,m,\mathrm{Pr}_1^M(c,k,b,m,\overline{s^{c,k}}))\\&=\mathrm{Pr}_1(a_k,k,a_m,m,\mathrm{Pr}_1(c,k,a_k,k,\overline{s^{c,k}}))=\mathrm{Pr}_1(a_k,k,a_m,m,\overline{s^k})=\overline{s^m}.\end{aligned}$$

This follows for the axioms of ψ_1.

So $\mathrm{Pr}_1(b,m,a_m,n,\bar{s})=\overline{s^m}=\mathrm{Pr}_1(b,m,a_m,m,\overline{s^{b_m,m}})$ since $|t_{b,m}|=|t_{a_m,m}|$, $\bar{s}=\overline{s^{b,m}}$. It thus follows that there are functions $s_1,\cdots s_l$ on $|M|$ such that for every n,b, if $a_n\subseteq b$ and $(G_1^{b,n})^M\neq\emptyset$ and $\overline{s^{b,n}}=\langle\langle s_1^*,s_1'^*\rangle,\cdots\langle s_l^*,s_l'^*\rangle\rangle$, then $\langle s_i^*,s_i'^*\rangle$ is the n-approximation induced by s_i on b. By the definition of G_1 $(M,s_1,\cdots,s_l)\models\forall x_1,\cdots x_{m_1}\chi_1(\bar{R},\bar{S},x_1,\cdots x_{m_1})$, a contradiction to the fact that $M\models\chi_4$.So M is rigid and Claim 5 is proved.

Proof of Theorem 1. We prove that ψ_4 has a rigid model of cardinality λ iff $\Sigma_{\kappa<\lambda}\kappa^{\aleph_0}=\lambda$ and $\lambda\models\chi$.

Suppose ψ_4 has a rigid model N of cardinality λ. Then $|P_0^N|=\|N\|=\lambda$ and P_0^N is countably closed, so $\Sigma_{\kappa<\lambda}\kappa^{\aleph_0}=\lambda$. By Claim 5, $P_0^N\models\chi_4$ so $\lambda\models\chi$.

Suppose $\lambda\models\chi$ and $\Sigma_{\kappa<\lambda}\kappa^{\aleph_0}=\lambda$, and let $r_1,\cdots r_{n_1}$ be relations of λ such that $(\lambda;r_1,\cdots r_{n_1})\models\chi_4$. Since $\lambda=\Sigma_{\kappa<\lambda}\kappa^{\aleph_0}$ there is a well-founded countably closed model of ψ_1 M whose universe is λ; then $N_4(M)$ is a rigid model of ψ_4 of cardinality λ.

Q.E.D.

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